

Quantum Gravity Momentum Representation and Maximum Invariant Energy

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Abstract

We use the idea of the symmetry between the spacetime coordinates x^μ and the energy-momentum p^μ in quantum theory to construct a momentum space quantum gravity geometry with a metric $s_{\mu\nu}$ and a curvature tensor $P^\lambda_{\mu\nu\rho}$. For a closed maximally symmetric momentum space with a constant 3-curvature, the volume of the p-space admits a cutoff with an invariant maximum momentum a . A Wheeler-DeWitt-type wave equation is obtained in the momentum space representation. The vacuum energy density and the self-energy of a charged particle are shown to be finite, and modifications of the electromagnetic radiation density and the entropy density of a system of particles occur for high frequencies.

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1 Introduction

The importance of the symmetry (reciprocity) between the spacetime coordinate operator \hat{x}^μ and the momentum operator \hat{p}^μ in quantum theory was pointed out by Born [1]. A free particle in quantum theory is described by a wave function

$$\psi = \exp\left(\frac{i}{\hbar} p_\mu x^\mu\right). \quad (1)$$

The wave function is completely symmetric in the two 4-vectors x^μ and p^μ . In a representation in Hilbert space of the operators \hat{x}^μ and \hat{p}^μ for which the \hat{x}^μ are diagonal, we have

$$\hat{p}^\mu \rightarrow (\hbar/i)\partial/\partial x^\mu, \quad (2)$$

while for diagonal \hat{p}^μ , we obtain

$$\hat{x}^\mu \rightarrow (-\hbar/i)\partial/\partial p^\mu. \quad (3)$$

A wave function in spacetime (x -space) can be Fourier transformed into another wave function in momentum space (p -space):

$$\phi(p) = \int d^4x \psi(x) \exp\left(\frac{i}{\hbar} p_\mu x^\mu\right). \quad (4)$$

When we consider gravitational phenomena, we picture the universe described by a spacetime geometry with the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad (5)$$

where $g_{\mu\nu}$ is the metric tensor. In classical physics, the momentum is describe by $m dx^\mu/d\tau$ where m is the test particle mass and τ is the proper time, and the transformation laws for p^μ are determined by x^μ . However, when we attempt to derive a quantum gravity theory, the particle motion is not described by a geodesic, but by a wave function and a wave equation. Following Born, we postulate a p -space line element

$$du^2 = s_{\mu\nu} dp^\mu dp^\nu \quad (6)$$

with the metric $s_{\mu\nu}$.

In analogy with the classical x -space geometry, we define the inverse p -space metric tensor $s^{\mu\nu}$ by

$$s^{\lambda\nu} s_{\mu\lambda} = \delta^\nu_\mu. \quad (7)$$

Moreover, we define a p -space curvature tensor

$$P^\nu{}_{\alpha\lambda\mu} = \frac{\partial L^\nu{}_{\alpha\lambda}}{\partial p^\mu} - \frac{\partial L^\nu{}_{\alpha\mu}}{\partial p^\lambda} - L^\nu{}_{\beta\lambda} L^\beta{}_{\alpha\mu} + L^\nu{}_{\beta\mu} L^\beta{}_{\alpha\lambda}, \quad (8)$$

where $L^\lambda{}_{\mu\nu}$ is the p -space connection

$$L^\lambda{}_{\mu\nu} = \frac{1}{2} s^{\lambda\sigma} \left(\frac{\partial s_{\mu\sigma}}{\partial p^\nu} + \frac{\partial s_{\nu\sigma}}{\partial p^\mu} - \frac{\partial s_{\mu\nu}}{\partial p^\sigma} \right). \quad (9)$$

We obtain from (8) the Ricci tensor

$$P_{\alpha\lambda} \equiv P^\nu{}_{\alpha\lambda\nu} = \frac{\partial L^\nu{}_{\alpha\lambda}}{\partial p^\nu} - \frac{\partial L^\nu{}_{\alpha\nu}}{\partial p^\lambda} - L^\nu{}_{\beta\lambda} L^\beta{}_{\alpha\nu} + L^\nu{}_{\beta\nu} L^\beta{}_{\alpha\lambda}. \quad (10)$$

The p-space possesses a diffeomorphism invariance in that we can define the transformation for an arbitrary contravariant p-vector:

$$A'^\mu = \frac{\partial p'^\mu}{\partial p^\alpha} A^\alpha, \quad (11)$$

and for a mixed p-space tensor such as $A^\mu{}_{\nu\lambda}$:

$$A'^\mu{}_{\nu\lambda} = \frac{\partial p'^\mu}{\partial p^\rho} \frac{\partial p^\sigma}{\partial p'^\nu} \frac{\partial p^\tau}{\partial p'^\lambda} A^\rho{}_{\sigma\tau}. \quad (12)$$

Our quantum gravity theory involving two quantum geometries, identified with the x-space and p-space geometries, leads naturally to two universal invariants, namely, the universal invariant value of the speed of light c and the invariant maximum momentum a (energy E_M). There have recently been interesting proposals to obtain two such universal constants, namely, ‘double special relativity’ and the kinematical structure of extended special relativity, based on a Born-Infeld electrodynamics with a hyperbolic complex structure [2, 3].

2 Relating Momentum Space Coordinates to Spacetime Coordinates

In quantum mechanics and relativistic quantum field theory there is a linear relation between spacetime and momentum space through a Fourier transform of a wave function or a field operator in flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is the flat spacetime metric. In the presence of a gravitational field, we lose this simple mapping between p-space and x-space.

In quantum field theory, it is convenient to perform calculations, such as Feynman diagrams, in the p-space representation. For closed loop diagrams these calculations are ultraviolet divergent, and in quantum gravity for expansions about flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(h^2), \quad (13)$$

in which the p-space and x-space position coordinates are related by Fourier transforms, the theory is not renormalizable due to the dimensional nature of Newton's constant G .

It seems unnatural to assume that position and momentum spaces are independent, because this would not lead to a simple flat spacetime limit of quantum theory or the gravitational theory based on the weak field expansion of the metric tensor about flat spacetime.

We postulate that there exists a transformation \mathcal{T} between x-space and p-space with the mapping

$$\phi(p) = \int d^4x \mathcal{T}[\psi(x), x^\mu p_\mu], \quad (14)$$

where \mathcal{T} is a matrix operator. The inverse transformation must also exist

$$\psi(x) = \int d^4p \tilde{\mathcal{T}}[\phi(p), x^\mu p_\mu]. \quad (15)$$

We also postulate the mapping of the metric tensor operators

$$\hat{g}_{\mu\nu}(x) = \int d^4p \mathcal{T}[\hat{s}_{\mu\nu}(p), x^\mu p_\mu], \quad (16)$$

and its inverse mapping. In the flat x-space and p-space limits, $g_{\mu\nu} = s_{\mu\nu} = \eta_{\mu\nu}$, we obtain the standard Fourier transform (4) and its inverse transformation.

We shall also postulate that for the transformations (14) and (15) there exists a transformation, \mathcal{U} , between x-space and p-space such that for diagonalized \hat{x}^μ in the Hilbert space of operators

$$\hat{p}^\mu \rightarrow \mathcal{U}\left(\frac{\hbar}{i} \frac{\partial}{\partial x^\mu}\right), \quad (17)$$

and for diagonalized \hat{x}^μ :

$$\hat{x}^\mu \rightarrow \mathcal{U}\left(\frac{-\hbar}{i} \frac{\partial}{\partial p^\mu}\right). \quad (18)$$

In the flat space limit these mappings reduce to the familiar ones (2) and (3).

3 Momentum Space Action Principle and Field Equations

We choose as our p-space action

$$S_p = \frac{1}{2\bar{\kappa}} \int d^4p \sqrt{-s} [P + 2\lambda_p] + S_c, \quad (19)$$

where $P = s^{\mu\nu} P_{\mu\nu}$ is the p-space scalar curvature, $s = \text{Det}(s_{\mu\nu})$, $\bar{\kappa}$ is a constant and λ_p and S_c are the p-space equivalents of the x-space cosmological constant and matter action, respectively.

The field equations obtained from the action are given by

$$P_{\mu\nu} - \frac{1}{2} s_{\mu\nu} P - \lambda_p s_{\mu\nu} = -\bar{\kappa} K_{\mu\nu}, \quad (20)$$

where $K_{\mu\nu}$ is the p-space equivalent of the x-space energy-momentum stress tensor $T^{\mu\nu}$:

$$K_{\mu\nu} = \frac{2}{\sqrt{-s}} \left(\frac{\delta S_c}{\delta s^{\mu\nu}} \right). \quad (21)$$

It satisfies the identities

$$\nabla_{p\nu} K^{\mu\nu} = 0. \quad (22)$$

Here, $\nabla_{p\nu}$ denotes the covariant derivative with respect to the p-space connection $L^\lambda_{\mu\nu}$.

The corresponding spacetime field equations are Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda_x g_{\mu\nu} = -\kappa T_{\mu\nu}, \quad (23)$$

where $\kappa = 8\pi G/c^4$, λ_x denotes the x-space cosmological constant and $T^{\mu\nu}$ satisfies the identities

$$\nabla_{x\nu} T^{\mu\nu} = 0, \quad (24)$$

where $\nabla_{x\nu}$ denotes the covariant derivative with respect to the metric $g_{\mu\nu}$.

4 Flat Momentum Space and the Momentum Space Null Cone

According to the quantum gravity reciprocity symmetry, we shall identify K^{00} as the 'density' of spacetime curvature per p-space 3-volume $V_{(3p)}$:

$$K^{00} = X(E, \mathbf{p}), \quad (25)$$

where $X(E, \mathbf{p})$ denotes the spacetime curvature density as a function of the energy E and the 3-momentum \mathbf{p} . When the p-space curvature $P^\lambda_{\mu\nu\rho} = 0$, then the p-space is ‘flat’ with the metric $s_{\mu\nu} = \eta_{\mu\nu}$ and the line element

$$du^2 = \left(\frac{dE}{c}\right)^2 - (dp_1^2 + dp_2^2 + dp_3^2). \quad (26)$$

We can now define transformations between p-space coordinates p_i ($i = 1, 2, 3$) and energy E as

$$E' = \frac{E - wp_1}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad (27)$$

$$p'_1 = \frac{p_1 - \left(\frac{w}{c^2}\right)E}{\sqrt{1 - \frac{w^2}{c^2}}}, \quad (28)$$

$$p'_2 = p_2, \quad (29)$$

$$p'_3 = p_3. \quad (30)$$

Here, w is the ‘relative speed’ of the primed and unprimed p-space frames. These transformations from the primed to the unprimed ‘p-frames’ leave the line element (26) unchanged: $du'^2 = du^2$. For the p-space null cone determined by $du^2 = 0$, we obtain

$$\frac{dE}{dp} = c, \quad (31)$$

where $p = |\mathbf{p}|$. The transformations (27)-(30) form the p-space homogeneous Lorentz group $SO_p(3, 1)$.

5 Maximally Symmetric Momentum Space Solution and a Maximum Invariant Momentum

We shall now assume that the p-space geometry is homogeneous and that the tensor density $K^{\mu\nu}$ is independent of the spatial momentum coordinates p^i , and that it can only depend on the energy E . We further assume that the p-space is isotropic on the large scale distribution of p-space points. The

mathematical expression of this postulate is that the 3-dimensional p-space will be a space of constant curvature:

$$P_{\lambda\mu\nu\rho} = C(s_{\lambda\nu}s_{\mu\rho} - s_{\lambda\rho}s_{\mu\nu}), \quad (32)$$

where $C = \text{constant}$. We apply this equation to the 3-dimensional subspace of the p-space, so we shall have

$$P_{iklm} = C(s_{il}s_{km} - s_{im}s_{kl}). \quad (33)$$

Contracting (33) with respect to s^{im} we get

$$P_{kl} = -2Cs_{kl}. \quad (34)$$

For a 3-space, Eq.(33) is equivalent to (34), so that the line element is spherically symmetric and each of the 3-dimensional points can be taken as the origin of the p-space coordinate system. The metric for constant 3-dimensional p-space curvature is

$$d\sigma_p^2 \equiv \gamma_{ik}dp^i dp^k = \frac{dp^2}{1 - \zeta\left(\frac{p^2}{a^2}\right)} + p^2(d\chi^2 + \sin^2 \chi d\xi^2), \quad (35)$$

where a is a constant invariant momentum and ζ has the values $+1, -1, 0$. For $\zeta = +1$ the p-space is a closed space of constant curvature. We shall choose $\zeta = +1$ so that there exists a *maximum* momentum a corresponding to a maximum invariant energy $E_M = ca$. The line element (35) is the p-space equivalent of the Friedmann-Robertson-Walker line element in 3-dimensional x-space:

$$d\sigma_x^2 = \frac{dr^2}{1 - \zeta\left(\frac{r^2}{\bar{r}^2}\right)} + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (36)$$

where \bar{r} is a constant.

The 3-volume of our p-space is given by

$$\int d^3p \sqrt{\gamma} = 4\pi \int dp \frac{p^2}{\sqrt{1 - \frac{p^2}{a^2}}}, \quad (37)$$

where $\gamma = \text{Det}(\gamma_{ik})$. This leads to the differential volume element

$$d\Omega_p = \frac{p^2 dp d\chi d\xi \sin \chi}{\sqrt{1 - \frac{p^2}{a^2}}}. \quad (38)$$

We can now introduce the p-space 4-velocity, $w^\mu = c^2 p^\mu / E$,¹ satisfying the condition

$$s_{\mu\nu} w^\mu w^\nu = 1. \quad (39)$$

Let us consider a fluid description of the p-space source tensor

$$K^{\mu\nu} = [X(E) + Z(E)] w^\mu w^\nu - Z(E) s^{\mu\nu}, \quad (40)$$

where $X(E)$ denotes the density of spacetime curvature per p-space 3-volume $V_{(3p)}$ and $Z(E)$ the elasticity of space. These quantities correspond reciprocally to the energy density ρ and pressure p , respectively, in the x-space representation of the perfect fluid energy-momentum tensor

$$T^{\mu\nu} = (\rho c^2 + p) u^\mu u^\nu - p g^{\mu\nu}, \quad (41)$$

where $u^\mu = dx^\mu / d\tau$.

We choose w^μ to satisfy the condition

$$w^\mu = (1, 0, 0, 0). \quad (42)$$

Then the 4-dimensional p-space metric line element has the form

$$du^2 = \left(\frac{dE}{c} \right)^2 - B^2(E) d\sigma_p^2, \quad (43)$$

where $d\sigma_p^2$ is given by (35) and $B(E)$ is a scale factor that depends on the energy E . The volume of 4-dimensional p-space is given by

$$\int d^4 p \sqrt{-s} = \frac{4\pi}{c} \int dE dp \frac{B^3(E) p^2}{\sqrt{1 - \frac{p^2}{a^2}}}. \quad (44)$$

There are two other maximally symmetric solutions to the p-space field equation, namely, the maximally symmetric de Sitter and anti-de Sitter solutions.

We can also consider the spherically symmetric energy independent solution of the p-space field equations (20) for $K_{\mu\nu} = \lambda_p = 0$:

$$P_{\mu\nu} = 0. \quad (45)$$

The line element is given by

$$du^2 = \left(\frac{dE}{c} \right)^2 \left(1 - \frac{2A}{p} \right) - \frac{dp^2}{1 - \frac{2A}{p}} + p^2 (d\chi^2 + \sin^2 \chi d\xi^2). \quad (46)$$

¹The Hamiltonian definition of the velocity is $v = dE/dp$.

Here, A is a constant of integration associated with the singularity at the origin of the p-space coordinates $p = 0$, corresponding to $2GM/c^2$ in the Schwarzschild solution in x-space. The solution has the asymptotically flat p-space boundary condition $s_{\mu\nu} \rightarrow \eta_{\mu\nu}$. We see that the line element exhibits a momentum space ‘event horizon’ at $p = 2A$, equivalent to the spacetime Schwarzschild black hole event horizon at $r_s = 2GM/c^2$.

6 Uncertainty Principle for Spacetime and Momentum Space Metrics

We shall postulate that our quantum gravity theory possesses an uncertainty principle for the x- and p-space metric operators $\hat{g}_{\mu\nu}$ and $\hat{s}_{\mu\nu}$:

$$\Delta\hat{g}_{\mu\nu}\Delta\hat{s}_{\rho\sigma} \geq \frac{\hbar}{a\ell}C_{\mu\nu\rho\sigma}, \quad (47)$$

where a denotes as before the maximum momentum, ℓ denotes a *minimum length* and

$$C_{\mu\nu\rho\sigma} = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} - \eta_{\mu\nu}\eta_{\rho\sigma}. \quad (48)$$

We then have the quantum commutation relation for the metric operators

$$[\hat{g}_{\mu\nu}, \hat{s}_{\rho\sigma}] = i\frac{\hbar}{a\ell}C_{\mu\nu\rho\sigma}\delta(x, p). \quad (49)$$

7 Momentum Space Wave Equation

We must obtain a wave equation for a particle to complete the p-space dynamics. To this end, we derive a p-space Hamiltonian formulation of gravity [4]. For our p-space compact manifold with boundary ∂M , we require that a variation of the metric $s_{\mu\nu}$ vanishes on ∂M but its normal derivative does not. Then we must add a surface term and the action becomes

$$S'_p = S_p + \frac{1}{\bar{\kappa}} \int d^3p \sqrt{f} F, \quad (50)$$

where F is the contraction of the extrinsic p-space curvature F_{ij} of the boundary 3-surface, and f is $\text{Det}(f_{ij})$ induced on the 3-surface.

The p-space metric is now given by

$$du^2 = \left(\frac{WdE}{c} \right) - f_{ij} \left[W^i \left(\frac{dE}{c} \right) + dp^i \right] \left[W^j \left(\frac{dE}{c} \right) + dp^j \right], \quad (51)$$

where W is the p-space lapse function and W^i is the shift function. The extrinsic curvature is

$$F_{ij} = \frac{1}{2W} \left[W_{i|k} + W_{k|i} - \dot{f}_{ij} \right], \quad (52)$$

where $|$ denotes covariant derivative with respect to f_{ij} and \dot{f}_{ij} denotes differentiation with respect to the energy E .

The variable conjugate to f_{ij} is

$$\Pi^{ij} \equiv \frac{\delta \mathcal{L}_p}{\delta \dot{f}_{ij}} = \frac{\sqrt{f}(F^{ij} - f^{ij}F)}{2\bar{\kappa}}, \quad (53)$$

where \mathcal{L}_p is the Lagrangian density associated with the action S_p . The Hamiltonian for a closed p-space geometry is given by

$$H_p = \int d^3p \left(\Pi^{ij} \dot{f}_{ij} + \Pi^i \dot{W}_i + \Pi \dot{W} - \mathcal{L}_p \right) = \int d^3p (W \mathcal{H}_p + W_i \mathcal{H}_p^i), \quad (54)$$

where

$$\mathcal{H}_p = \frac{\sqrt{f}(F_{ij}F^{ij} - F^2 - P^{(3)})}{2\bar{\kappa}} = 2\bar{\kappa}Q_{ijkl}\Pi^{ij}\Pi^{kl} - \frac{\sqrt{f}P^{(3)}}{2\bar{\kappa}}. \quad (55)$$

Here, $P^{(3)}$ is the 3-curvature and

$$Q_{ijkl} = \frac{1}{2\sqrt{f}}(f_{ik}f_{jl} + f_{il}f_{jk} - f_{ij}f_{kl}). \quad (56)$$

We have two primary constraints

$$\Pi \equiv \frac{\delta \mathcal{L}_p}{\delta \dot{W}} = 0, \quad \Pi^i \equiv \frac{\delta \mathcal{L}_p}{\delta \dot{W}_i} = 0. \quad (57)$$

Because $\delta H/\delta W = \delta H/\delta W_i = 0$, we have the secondary constraints

$$H_p = H_p^i = 0. \quad (58)$$

We now define a p-space wave function $\Psi[f_{ij}]$ and obtain the wave equation

$$\left[\frac{Q_{ijkl}}{(2\bar{\kappa})^2} \frac{\delta}{\delta f_{ij}} \frac{\delta}{\delta f_{kl}} + \frac{\sqrt{f}(P^{(3)} - 2\lambda_p)}{2\bar{\kappa}} - K^0_0 \right] \Psi[f_{ij}, \phi] = 0, \quad (59)$$

where ϕ denotes matter fields. This is the p-space equivalent of the Wheeler-DeWitt equation in x-space [5]. The wave function associated with the Wheeler-DeWitt equation in x-space does not depend on the time t . Equivalently, for the reciprocity symmetry we have postulated, the wave function $\Psi[f_{ij}, \phi]$ does not depend on the energy E but only on the 3-geometry f_{ij} and the matter fields ϕ . Likewise, this means that the role of energy in the p-space geometry is unclear.

8 Applications of Invariant Maximum Momentum

Let us consider an electromagnetic radiation field with a vector potential $A_\mu = (\mathbf{A}, U)$ in a small localized region of spacetime, which is approximately flat $g_{\mu\nu} \sim \eta_{\mu\nu}$. Then, the Fourier series representations of the scalar and vector potentials U and \mathbf{A} are

$$U = c \left(\frac{8\pi}{V_{(3x)}} \right)^{1/2} \sum_s Q_s(t) \cos(\Theta_s), \quad (60)$$

and

$$\mathbf{A} = c \left(\frac{8\pi}{V_{(3x)}} \right)^{1/2} \sum_s B_s(t) \sin(\Theta_s), \quad (61)$$

where $V_{(3x)}$ is the x-space 3-volume and

$$\Theta_s = \frac{2\pi\nu_s}{c}(\mathbf{n}_s, \mathbf{x}) + \beta_s. \quad (62)$$

Here, \mathbf{n}_s is a unit vector giving the direction of the standing wave and β_s is a constant.

In our quantum gravity momentum representation there is an upper limit to the momentum, $p = a$. We assume that in an approximately flat spacetime this remains true. This means that for a quantum system of independent

particles the increment number density of quantum states of weight f in a momentum element $d\Omega_p$ is, according to (38), given by [1]:

$$dn = \left(\frac{f}{h^3}\right)d\Omega_p = \left(\frac{f}{h^3}\right)\frac{d^3p}{\sqrt{\left(1 - \frac{p^2}{a^2}\right)}}. \quad (63)$$

Because of the square root there is a maximum number of allowed states in a quantum system. Thus, the total number density of quantum states is *finite*:

$$n = \left(\frac{f}{h^3}\right) \int \frac{d^3p}{\sqrt{\left(1 - \frac{p^2}{a^2}\right)}} = \left(\frac{4\pi f a^3}{h^3}\right) \int_0^1 \frac{dy y^2}{\sqrt{(1 - y^2)}} = \frac{f\pi^2 a^3}{h^3}. \quad (64)$$

Let us consider the zero-point energy associated with a system of oscillators. For stationary states we have

$$E_r = \sum_s h\nu_s \left(n_s + \frac{1}{2}\right). \quad (65)$$

By using the formula (64) with $f = 2$, the zero-point vacuum energy is given by

$$\begin{aligned} \rho_{\text{vac}} &= \sum_s \frac{1}{2} h\nu_s = \frac{c}{2} \sum_s p_s = \frac{4\pi c}{h^3} \int_0^{a_v} \frac{dp p^3}{\sqrt{\left(1 - \frac{p^2}{a_v^2}\right)}} \\ &= \frac{4\pi c a_v^4}{h^3} \int_0^1 \frac{dy y^3}{\sqrt{(1 - y^2)}} = \frac{8\pi c a_v^4}{3h^3}. \end{aligned} \quad (66)$$

We see that the zero-point vacuum density of oscillators is finite, and its magnitude is determined by the momentum scale a_v .

The modified energy density of radiation takes the form

$$\mathcal{E} = \int_0^{1/b} d\nu \mathcal{U}(\nu, T). \quad (67)$$

Here, we have

$$\mathcal{U}(\nu, T) = \frac{8\pi h\nu^3}{c^3 \left(\exp(h\nu/kT) - 1\right) \sqrt{(1 - (\nu b)^2)}}, \quad (68)$$

where $p = h\nu/c$ and $b = h/ac$. Eq.(67) gives for $b = 0$ ($a = \infty$) the result

$$\mathcal{E} = a_B T^4, \quad (69)$$

where $a_B = 7.56 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$. Eq.(69) is the Stefan-Boltzmann law. We see that at very high momentum (frequency) $p \sim a$ the Planck radiation density and the Stefan-Boltzmann law are modified by the magnitude of the maximum momentum a . Such a change at high frequencies could possibly be detected in the CMB Planck spectrum.

For the entropy density of a system of particles with temperature T , we obtain

$$\begin{aligned} \mathcal{S} = \left(\frac{8\pi}{c^3 T} \right) \int_0^{1/b} \frac{d\nu}{\sqrt{(1 - (\nu b)^2)}} & \left\{ kT \ln[(1 - \exp(-h\nu/kT))^{-1}] \nu^2 \right. \\ & \left. + \frac{h\nu^3}{(\exp(h\nu/kT) - 1)} \right\}. \end{aligned} \quad (70)$$

By making the substitution $y = b\nu$ this becomes

$$\begin{aligned} \mathcal{S} = \left(\frac{8\pi}{c^3 T} \right) \int_0^1 \frac{dy}{\sqrt{(1 - y^2)}} & \left\{ \left(\frac{kT}{b^3} \right) \ln[(1 - \exp(-hy/bkT))^{-1}] y^2 \right. \\ & \left. + \left(\frac{h}{b^4} \right) \frac{y^3}{[\exp(hy/bkT) - 1]} \right\}. \end{aligned} \quad (71)$$

The total entropy for a system of particles is finite.

For the Coulomb energy associated with charged particles, we get

$$E_c = \left(\frac{h^2}{\pi V_{(3x)}} \right) \sum_{k,l} e_k e_l R_{kl}, \quad (72)$$

where e_k is the charge of the k th particle and

$$R_{kl} = \sum_s \left(\frac{\cos(\Theta_{sk}) \cos(\Theta_{sl})}{p_s^2} \right) = \frac{V_{(3x)}}{h^3} \int \frac{d^3 p}{p^2 \sqrt{\left(1 - \frac{p^2}{a^2}\right)}} \cos(\Theta_k) \cos(\Theta_l). \quad (73)$$

By taking the mean value of $\cos(\Theta_k) \cos(\Theta_l)$ over all directions of propagation and phases, we find

$$\begin{aligned}
R_{kl} &= \left(\frac{\pi V_{(3x)}}{h^3} \right) \int_0^a \int_{-1}^1 \frac{dp d\mu \cos\left(2\pi p \mu r_{kl}/h\right)}{\sqrt{\left(1 - \frac{p^2}{a^2}\right)}} \\
&= \frac{V_{(3x)}}{h^2 r_{kl}} \int_0^a \frac{dp \sin\left(2\pi p r_{kl}/h\right)}{p \sqrt{\left(1 - \frac{p^2}{a^2}\right)}}, \tag{74}
\end{aligned}$$

where $r_{kl} = r_k - r_l$ denotes the distance between the charges e_k and e_l .

We introduce

$$f(q) = \frac{2}{\pi} \int_0^1 \frac{dy}{\sqrt{(1-y^2)}} \frac{\sin(yq)}{y} = \int_0^q dz J_0(z), \tag{75}$$

where $J_0(z)$ is the Bessel function. Substituting $f(q)$ into (74) gives

$$R_{kl} = \frac{\pi V_{(3x)}}{2h^2} \frac{1}{r_{kl}} f\left(2 \frac{r_{kl}}{r_0}\right), \tag{76}$$

where

$$r_0 = \frac{h}{\pi a}. \tag{77}$$

Inserting (76) into (72), we arrive at the modified Coulomb energy [1]:

$$E_c = \frac{1}{2} \sum_{k,l} \frac{e_k e_l}{r_{kl}} f\left(2 \frac{r_{kl}}{r_0}\right). \tag{78}$$

We have $f(q) \rightarrow 1$ for $q \rightarrow \infty$ and $f(q)/q \rightarrow 1$ for $q \rightarrow 0$, so that we retain the classical Coulomb energy for $r_{kl} \gg r_0$ and a *finite* self-energy of a charged particle for $r \rightarrow 0$:

$$E_c = \frac{e^2}{r_0}. \tag{79}$$

This result is similar to the regularization of Coulomb's law in Born-Infeld electrodynamics [6].

The result that a maximum momentum a leads to a regularization of Coulomb's law in electrodynamics, leads one to believe that a similar regularization of the Schwarzschild singularity in the spacetime Schwarzschild solution of Einstein gravity could be realized in quantum gravity theory.

9 Conclusions

We have proposed that the symmetry between the two vector operators \hat{p}^μ and \hat{x}^μ in quantum theory is a fundamental property of nature, that should be exploited in a quantum gravity theory. Two metric tensor operators $\hat{g}_{\mu\nu}$ and $\hat{s}_{\mu\nu}$ associated, respectively, with the geometries of spacetime (x-space) and momentum space (p-space) are introduced with their respective pseudo-Riemannian geometries. Field equations for both geometries are postulated with the x-space equations being the Einstein field equations with an energy-momentum tensor density $T_{\mu\nu}$, while the p-space field equations are associated with a tensor density $K_{\mu\nu}$, identified with the density of spacetime curvature reciprocal to the energy-momentum density in spacetime. The absence of spacetime curvature produces a flat p-space geometry. This suggests that the existence of a curved spacetime manifold is associated with a curvature of the momentum p-space.

By assuming a closed, homogeneous and isotropic p-space geometry, we find that the volume of the space has a maximum invariant momentum (energy) a (E_M), which leads to a finite statistical number density of quantum states. This is in contrast to the spacetime volume, which for cosmological scales can be open and infinite. Thus, the quantum gravity momentum space geometry leads to a natural invariant, ultraviolet cutoff, E_M , for *all particle interactions* in a closed particle system. We apply this result to the calculation of the vacuum density ρ_{vac} giving a finite value. We also find that the self-energy of a charged particle is regularized.

The quantum gravity theory proposed leads to an uncertainty principle for the two reciprocal metric tensor operators $\hat{g}_{\mu\nu}$ and $\hat{s}_{\mu\nu}$, involving the maximum momentum a and a minimum length ℓ . The momentum representation geometry lends itself best to describing microscopic particle physics and quantum gravity, while the spacetime representation geometry describes the macroscopic properties of the universe. Both representation geometries complement one another and describe the quantum and large scale properties of the universe.

For a compact p-space the x-space at short distances will be discrete and described by a lattice structure with a minimum length ℓ , which we can identify with the Planck length $\ell = \ell_{PL}$, where $\ell_{PL} = \sqrt{\hbar G/c^3}$.² In order

²A discrete lattice structure of spacetime has been promoted as a basis for quantum gravity [7].

to preserve local Lorentz symmetry, the discrete spacetime corresponds to a noncommutative geometry [8]. For long wave-length gravity using our space-time metric $g_{\mu\nu}$, we must assume a limit exists that gives us the macroscopic continuum for distances much greater than ℓ_{PL} .

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